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Quark number fractionalization in $N = 2$ supersymmetric $SU(2) \times U(1)^{N_f}$ gauge theories

Giuseppe Carlino and Kenichi Konishi
Dipartimento di Fisica – Università di Genova
Istituto Nazionale di Fisica Nucleare – Sezione di Genova
Via Dodecaneso, 33 – 16146 Genova (Italy)
E-mail: konishi@infn.ge.infn.it

Haruhiko Terao
Department of Physics, Kanazawa University
Kanazawa, Japan
E-mail: terao@hep.s.kanazawa-u.ac.jp

Abstract: Physical quark-number charges of dyons are determined, via a formula which generalizes that of Witten for the electric charge, in $N = 2$ supersymmetric theories with $SU(2) \times U(1)^{N_f}$ gauge group. The quark numbers of the massless monopole at a nondegenerate singularity of the space of vacua turn out to vanish in all cases. A puzzle related to CP invariant cases is solved. Generalization of our results to $SU(N_c) \times U(1)^{N_f}$ gauge theories is straightforward.

Keywords: Solitons Monopoles and Instantons, Nonperturbative Effects, Duality in Gauge Field Theories, Supersymmetry and Duality.

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Contents

1. Introduction

The breakthrough achieved in the celebrated works of Seiberg and Witten [1, 2] has made possible, for the first time, to go beyond the semiclassical quantization in the study of soliton dynamics in non Abelian gauge theories in four dimensions.[3] Of particular interest among their properties is the electric and quark-number fractionalization. In Ref. [4] it was shown that, in the semiclassical limit, the exact Seiberg-Witten prepotentials and mass formula reproduce these effects correctly, in accordance with the standard semiclassical calculations [5, 6, 7]. In the case of the electric charge fractionalization one has here an exact quantum result valid even where the semiclassical approximation breaks down.

The situation for the quark-number fractionalization is somewhat different. As long as the $U(1)$ symmetries associated with the quark numbers are global, the "physical quark number of monopoles" is a somewhat obscure concept[6], even though such a quantum number is conserved and not spontaneously broken. This is due to the fact that there are no local fields within the theory coupled to the conserved quark-number currents J_i^μ. Indeed, the correlation functions

$$\Pi^{\mu\nu}(Q) = i \int d^4x\, e^{-iQx}\, \langle T\{J_i^\mu(x)\, J_i^\nu(0)\}\rangle \tag{1.1}$$

cannot be easily analyzed at low energies, although at high energies these can be computed perturbatively due to asymptotic freedom.

In this paper, we consider $N = 2$ supersymmetric $SU(2) \times U(1)^{N_f}$ gauge theories ($N_f = 1, 2, 3$) where the Abelian factors correspond to the conserved quark numbers; more precisely we consider these theories in the limit

$$g_i \to 0+\,, \tag{1.2}$$

g_i being the $U_i(1)$ coupling constant. In other words, we introduce the hypothetical weak $U(1)^{N_f}$ gauge bosons and their $N = 2$ partners in order to probe the strong interaction dynamics, which is dominated by the $SU(2)$ interactions. This theory has a large vacuum degeneracy parametrized by $N_f + 1$ moduli parameters,

$$u = \langle \mathrm{Tr}\, \Phi^2 \rangle; \quad a_i = \langle A_i \rangle = m_i/\sqrt{2},\ (i = 1, 2, \ldots N_f)\,. \tag{1.3}$$

The physical i-th quark number (charge) S_i of a given particle is *by definition* its coupling strength to the $U_i(1)$ gauge boson, measured in the unit of the coupling constant, g_i. g_i is common to all particles (elementary and solitonic) and depends only on the scale whereas the charge S_i depends on the particle considered. For an elementary particle, say the j-th quark, $S_i = \delta_{ji}$. The latter is not renormalized, as is well known. Our main aim is to determine the value of S_i for a given dyon, in each vacuum $(u, a_1, a_2, \ldots, a_{N_f})$.

2. Fractional quark numbers of dyons as boundary effects

The theory is described by the Lagrangian,

$$\begin{aligned} L = {} & \frac{1}{8\pi} \mathrm{Im}\, \tau_{cl} \left[\int d^4\theta\, \Phi^\dagger e^V \Phi + \int d^2\theta\, \frac{1}{2} W W \right] + \\ & + \sum_{i=1}^{N_f} \frac{1}{8\pi} \mathrm{Im}\, \tau_i \left[\int d^4\theta\, A_i^\dagger A_i + \int d^2\theta\, \frac{1}{2} W_i W_i \right] + L^{(quarks)}\,, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} L^{(quarks)} = {} & \sum_i \Big[\int d^4\theta\, \{ Q_i^\dagger e^V Q_i\, e^{V_i} + \tilde{Q}_i e^{-V} \tilde{Q}_i^\dagger\, e^{-V_i} \} \\ & + \int d^2\theta \sqrt{2} \{ \tilde{Q}_i \Phi Q^i + A_i \tilde{Q}_i Q^i \} + h.c. \Big]\,, \end{aligned} \tag{2.2}$$

where $\{\Phi, W\}$ and $\{A_i, W_i\}$ are $N = 2$ vector supermultiplets containing the gauge bosons. Classically this theory has the flat directions parametrized by the vevs eq. (1.3). In a generic point of such vacuum moduli space, the gauge symmetry is broken to $U(1) \times U(1)^{N_f}$ and the low energy effective Lagrangian must describe $N_f + 1$ massless gauge bosons and their superpartners, and eventually, also light quarks or dyons. Its form is restricted by $N = 2$ supersymmetry to be

$$L^{(eff)} = \frac{1}{8\pi} \mathrm{Im} \sum_{i,j=0}^{N_f} \int d^2\theta\, \frac{1}{2} \tau_{ij} W_i W_j + \sum_{i=0}^{N_f} \int d^4\theta\, \frac{\partial F}{\partial A_i} \bar{A}_i + L^{(light)}, \tag{2.3}$$

where $\tau_{ij} = \frac{\partial^2 F}{\partial A_i \partial A_j}$, and $L^{(light)}$ describes either light quarks or light dyons. $F(a_0, \{a_i\}; \Lambda; \{\Lambda_i\})$ is the prepotential, holomorphic in its arguments. Also, we introduced a notation

$$A_0 \equiv A, \quad V_0 \equiv V, \qquad A_{D0} \equiv A_D, \quad V_{D0} \equiv V_D, \tag{2.4}$$

to indicate the vector multiplet related to the original $SU(2)$ gauge multiplet. Λ_i is the position of the Landau pole associated to the i-th $U(1)$ gauge interaction.

The form of $L^{(light)}$ near one of the quark singularities (when $u = m_i^2 \gg \Lambda^2$), is fixed since the quantum numbers of the light quarks with respect to the $U(1) \times U(1)^{N_f}$ gauge group are known.

$N_f = 1$

name	n_1	n_m	n_e	$SO(2)$	θ_{eff}
M	0	1	0	$+$	0
M'	1	1	1	$-$	$-\pi$
M''	0	1	2	$+$	-2π

$N_f = 2$

name	n_1	n_2	n_m	n_e	$SO(4)$	$SU(2) \times SU(2)$	θ_{eff}
M_1	0	0	1	0	$+$	$(\mathbf{2},\ \mathbf{1})$	0
M_2	1	1	1	0	$+$	$(\mathbf{2},\ \mathbf{1})$	0
M_1'	1	0	1	1	$-$	$(\mathbf{1},\ \mathbf{2})$	$-\pi$
M_2'	0	1	1	1	$-$	$(\mathbf{1},\ \mathbf{2})$	$-\pi$

$N_f = 3$

name	n_1	n_2	n_3	n_m	n_e	$SO(6)$	$SU(4)$	θ_{eff}
M_0	0	0	0	1	0	$+$	$\mathbf{4}$	0
M_1	1	1	0	1	0	$+$	$\mathbf{4}$	0
M_2	1	0	1	1	0	$+$	$\mathbf{4}$	0
M_3	0	1	1	1	0	$+$	$\mathbf{4}$	0
N	0	0	0	2	1		$\mathbf{1}$	$-\pi/2$

Table 1: Classical quark number charges and other global quantum numbers of light dyons. $\pm$ denotes the $SO(2N_f)$ chirality of the spinor representation. θ_{eff} gives the value of the effective θ parameter where the corresponding dyon becomes massless.

On the other hand, the monopoles acquire the quark numbers dynamically. Semi-classically they arise through the zero modes of the Dirac Hamiltonian in the monopole background [8, 3, 2, 4]. The *classical* quark numbers of the dyons, which become massless at various singularities of QMS are given in Table 1.

Note that, in contrast to Ref. [4] we have chosen the *classical* quark number charges for monopoles to start with. [1] The dynamical, fractional part of quark number charges can be determined as follows.

The structure of the low energy effective Lagrangian, when one of these monopoles is light (near one of the singularities in the u plane), is then fixed by their *integer* quantum numbers, n_m, n_e and

$$n_i \equiv S_i^{(cl)}, \quad i = 1, 2, \dots N_f \,. \tag{2.5}$$

Near the singularity of QMS where a $(n_m, n_e, \{n_i\})$ dyon is light $L^{(light)}$ has the form (assuming there is only one such dyon)

$$\begin{aligned} L^{(light)} = & \int d^4\theta \, [M^\dagger e^{n_m V_{0D} + n_e V_0 + \sum n_i V_i} M + \tilde{M}^\dagger e^{-n_m V_{0D} - n_e V_0 - \sum n_i V_i} \tilde{M}] + \\ & + \int d^2\theta \, \sqrt{2} (n_m A_{0D} + n_e A_0 + \sum n_i A_i) \tilde{M} M + h.c. \end{aligned} \tag{2.6}$$

The fact that the monopole is coupled to the weak $U(1)$ gauge fields with the (apparent) integer charges, does not mean that its physical charges are equal to the classical ones. The point is that there are nontrivial boundary effects to be taken into account, just as the Witten's effect for the electric charge of the monopole, in the presence of the θ term, $(\theta/32\pi^2) F_{\mu\nu} \tilde{F}^{\mu\nu}$ [9, 10].

In our case, the crucial term is the mixed gauge kinetic term, $\tau_{0i} W_0 W_i$ of eq. (2.3). In fact, this term yields a term in the energy

$$\frac{1}{4\pi} \mathrm{Re}\, \tau_{0i} \int d^3x \, \mathbf{E}_i \cdot \mathbf{H}_0 \,, \tag{2.7}$$

where $\mathbf{E}_i$ and $\mathbf{H}_0$ stand respectively for the "electric" field associated with the weak, quark number $U_i(1)$ and for the "magnetic" field associated with the strong $U_0(1)$ (related to the $SU(2)$) gauge interactions. In the presence of a static monopole,

$$\int d^3x \, \mathbf{E}_i \cdot \mathbf{H}_0 \simeq \int d^3x \, (-\nabla \phi_i(x)) \cdot \nabla \frac{n_m}{r} = -4\pi n_m \int d^3x \, \phi_i(x) \delta^3(\mathbf{x}) \,, \tag{2.8}$$

hence eq. (2.7) implies that the magnetic monopole, when observed at spatial infinity, possesses an additional quark number charge,

$$\Delta S = n_m \, \mathrm{Re}\, \tau_{0i} = n_m \, \mathrm{Re} \frac{\partial^2 F}{\partial a \partial a_i} \,. \tag{2.9}$$

The true, physical i-th quark number charge of such a dyon is therefore given by

$$S_i^{(phys)} = n_i + n_m \, \mathrm{Re}\, \tau_{0i}. \tag{2.10}$$

This, which generalizes Witten's well-known formula [9], is our main result.

[1] This is, strictly speaking, unnecessary. One can start with any choice of S and adjust the constant part of a_D proportional to quark masses accordingly, as explained in [4]. The final result for the physical quark number is the same, whatever initial choice for S one makes, but the final formulas look most elegant with our choice.

3. Generalization to $SU(N_c)$

The fractional quark numbers of dyons in $N=2$ supersymmetric $SU(N_c)$ gauge theories [11] can be found by similar considerations. In the QMS the $SU(N_c)$ gauge group is broken to $U(1)^{N_c}$. The dyon carries magnetic and electric charges of each unbroken $U(1)$ and the associated quantum numbers are denoted by $(n_m^r, n_e^r), (r=1,\cdots,N_c)$. The gauge couplings in the low energy effective action are also generalized to τ_{rs}, $(r,s=1,\cdots,N_c)$, with the mixed θ terms, $(\theta_{rs}/32\pi^2)F^r_{\mu\nu}\tilde{F}^{s\mu\nu}$. Therefore the r-th physical electric charge Q^r is found to be

$$Q^r = n_e^r + \sum_{s=1}^{N_c} \mathrm{Re}\,\tau_{rs}\; n_m^s\,. \tag{3.1}$$

In order to define the physical quark numbers unambiguously we consider $SU(N_c)\times U(1)^{N_f}$ gauge theories. The $U(1)^{N_c}\times U(1)^{N_f}$ effective Lagrangian is given by

$$L^{(eff)} = \frac{1}{8\pi}\mathrm{Im}\sum_{I,J}\int d^2\theta\,\frac{1}{2}\tau_{IJ}W_IW_J + \sum_I \int d^4\theta\, A_{DI}\bar{A}_I + L^{(light)}\,, \tag{3.2}$$

where I,J denote the combined suffix $(r,i),(r=1,\cdots,N_c, i=1,\cdots,N_f)$. From the mixed couplings between the "color" $U(1)$ and the "flavor" $U(1)$ field strengths we find the physical i-th quark number of a dyon in $SU(N_c)$ gauge theories as

$$S_i^{(phys)} = n_i + \sum_{r=1}^{N_c}\mathrm{Re}\,\tau_{ri}\; n_m^r. \tag{3.3}$$

4. Minimal coupling

One might wonder whether, having an "exact low energy effective Lagrangian" at hand, such fractional quark number charges should not appear as part of the standard minimal interaction terms. In fact, it is possible to interpret our result this way. In the case of Witten's effect this was pointed out in [12].

There is indeed a large class of arbitrariness in the choice of "dual" variables $A_D \equiv A_{0D}$, $V_D \equiv V_{0D}$, corresponding to a shift of these variables by terms linear in A, and A_i, V, and V_i, $i=1,2,\ldots,N_f$. These make a subgroup of $Sp(2+2N_f,R)$ which leaves eq. (2.3) form invariant (see however below). Actually, since the quark-number $U(1)$ groups are only weakly gauged, we can exclude those elements of $Sp(2+2N_f,R)$ which transform A_i $(i=1,2,\ldots,N_f)$ to their duals. In such a case, the most general form of the relevant subgroup of $Sp(2+2N_f,R)$ have been recently found by Alvarez-Gaumé et. al. [13]; they have the following general form:

$$\begin{pmatrix} A_D \\ A \\ A_{iD} \\ A_i \end{pmatrix} \to \begin{pmatrix} \alpha A_D + \beta A + p_i A_i \\ \gamma A_D + \delta A + q_i A_i \\ A_{iD} + p_i(\gamma A_D + \delta A) - q_i(\alpha A_D + \beta A) - p_i q_i \\ A_i \end{pmatrix}, \tag{4.1}$$

where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,R)$ and p_i, q_i are real.[2]

The transformations relevant to us are the ones with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ and p_i. These transformations leave the effective Lagrangian eq. (2.3) invariant, *except for* the shift,

$$\tau \to \tau + \beta; \quad \tau_{0i} \to \tau_{0i} + p_i \,. \tag{4.2}$$

Therefore all of $\mathrm{Re}\,\tau$, $\mathrm{Re}\,\tau_{0i}$ can be eliminated by an appropriate such transformation, i.e., by choosing $\beta = \mathrm{Re}\,\tau$, $p_i = \mathrm{Re}\,\tau_{0i}$. However $N=2$ supersymmetry imposes a simultaneous shift of vector superfields

$$V_D \to V_D + \beta V + p_i V_i; \quad V \to V; \quad V_i \to V_i \,; \tag{4.3}$$

in the effective Lagrangian involving light dyons such as eq. (2.6). The net result is that the real part in the coefficients of the mixed kinetic terms discussed previously has disappeared and, at the same time, the dyon with integer quantum numbers (n_m, n_e, n_i) is now coupled minimally to the vector fields A_μ and $A_{i\mu}$ with charges

$$Q_e = n_e + n_m \,\mathrm{Re}\,\tau; \quad \text{and} \quad S_i^{(phys)} = n_i + n_m \,\mathrm{Re}\,\tau_{0i} \,. \tag{4.4}$$

The first is Witten's effect ($\mathrm{Re}\,\tau = \theta_{eff}/\pi$), the second is our result.

The apparently local effective Lagrangian eq. (2.3) (with eq. (2.6)) with a nontrivial boundary effect, has been transformed by (4.3) into an explicitly nonlocal Lagrangian. Such an equivalence is to be expected after all, in view of the dyonic nature of our monopoles.

Our formula for the physical quark number eq. (4.4) can be expressed in terms of known quantities. Note that in the limit of weak $U(1)^{N_f}$ couplings the low energy effective coupling and θ parameter of the $SU(2)$ sector as well as the mass formula must be the same as in [2]. It means that the prepotential is essentially that given in [2]:

$$F(a_0, \{a_i\}; \Lambda; \{\Lambda_i\}) = F^{(SW)}(a_0, \{m_i\}; \Lambda)|_{m_i = \sqrt{2} a_i} + \sum_{i=1}^{N_f} C_i a_i^2 \,, \tag{4.5}$$

where possible terms linear in a and a_i have been dropped.[3] The last term contains the genuine free parameters of the theory, the $U_i(1)$ coupling constants at a given scale, or

[2]In [13] models with $N=2$ dilaton and mass "spurion" fields are studied and this leads them to consider a $Sp(4+2N_f, R)$ group. Here we restrict ourselves to renormalizable theories: this leaves only the quark masses to be replaced by the $N=2$ mass "spurion" fields in their language. The latter is equivalent to gauging the N_f quark-number $U(1)$ groups, as formulated here. Note that we use a slightly different notation from [13], a_i, a_{iD} instead of m_i, m_D^i, etc. Note that p_i, q_i are any *real* numbers, rational or not.

[3]The low energy effective action involves the second or higher derivatives of the prepotential. The term linear in a does affect the constant part of a_D which should be fixed by appropriate convention so that the mass formula is obeyed.

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the position of the corresponding Landau poles, $\{\Lambda_i\}$. Although these affect the $U_i(1)$ coupling constants g_i, they do not enter the calculation of the corresponding *charges*, eq. (2.10).

τ_{0i} can now be expressed as

$$\tau_{0i} = \frac{\partial^2 F}{\partial a \partial a_i} = \frac{\partial a_D}{\partial a_i}\Big|_a = \frac{\partial a_{iD}}{\partial a}\Big|_{a_i}, \quad \text{where} \quad a_D = \frac{\partial F}{\partial a} = \frac{\partial F^{(SW)}(a, \{m_i\}; \Lambda)}{\partial a}. \tag{4.6}$$

The partial derivative of a_D with respect to a_i can be further rewritten as

$$\tau_{0i} = \frac{\partial a_D}{\partial a_i}\Big|_a = \frac{\partial a_D}{\partial a_i}\Big|_u - \frac{d a_D}{da}\frac{\partial a}{\partial a_i}\Big|_u = \frac{\partial a_D}{\partial a_i}\Big|_u - \tau \frac{\partial a}{\partial a_i}\Big|_u . \tag{4.7}$$

The fractional quark charge can now be computed by using the known exact solution for

$$\frac{d a_D}{du} = \oint_\alpha \omega, \quad \frac{da}{du} = \oint_\beta \omega, \quad a_D = \oint_\alpha \lambda_{SW}, \quad a = \oint_\beta \lambda_{SW} \tag{4.8}$$

and their derivatives with respect to $a_i = m_i/\sqrt{2}$. The meromorphic differential λ_{SW}, related to ω by $\omega = \frac{\sqrt{2}}{8\pi}\frac{dx}{y} = \frac{\partial \lambda_{SW}}{\partial u}$, is given explicitly in [2, 13, 4, 14].

5. Riemann bilinear relation

An equivalent alternative formula for τ_{0i} can be found by first rewriting the formula (4.7) as

$$\tau_{0i} = \left(\frac{\partial a_D}{\partial a_i}\Big|_u \frac{\partial a}{\partial u}\Big|_{a_i} - \frac{\partial a_D}{\partial u}\Big|_{a_i} \frac{\partial a}{\partial a_i}\Big|_u \right) \Big/ \frac{\partial a}{\partial u}\Big|_{a_i} . \tag{5.1}$$

In terms of a meromorphic differential $\phi_i = \partial \lambda_{SW}/\partial a_i$, and the holomorphic differential ω, this can be written, by using Riemann bilinear relation [15], as

$$\tau_{0i} = \left(\oint_\alpha \phi_i \oint_\beta \omega - \oint_\alpha \omega \oint_\beta \phi_i \right) \Big/ \oint_\beta \omega = 2\pi i \sum_n \mathrm{Res}_{x_n^+} \phi_i \int_{x_n^-}^{x_n^+} \omega \Big/ \oint_\beta \omega , \tag{5.2}$$

where x_n^+ and x_n^- denote the poles of ϕ_i in the first and the second Riemann sheet respectively. The contour from x_n^- to x_n^+ must be taken so as to go around the branch point which is not encircled by the α-cycle. The positions of the poles x_n's (which are nontrivial for $N_f = 3$) are explicitly given in [13].

6. Semiclassical limit

As a first check of our result consider the semiclassical limit, $u \gg \Lambda^2$, $a_i = m_i/\sqrt{2} \gg \Lambda$. In this limit eq. (2.10) must reduce to the known result [5, 6]:

$$S_i^{(phys)} \simeq n_i + \frac{n_m}{2\pi} \mathrm{Arg} \frac{a + m_i/\sqrt{2}}{m_i/\sqrt{2} - a} . \tag{6.1}$$

which is indeed the case as can be verified by a direct calculation, similar to the one in the Appendix C of [4]. [4]

7. Vanishing quark numbers of massless monopoles

An interesting special case is that of the quark number charges of a massless dyon, at one of the singularities of QMS. Since such an object occurs in a theory in which the original $SU(2)$ coupling constant becomes large in the infrared, the semiclassical method does not apply there.

For simplicity we consider the cases in which the singularity is nondegenerate, with only one massless monopole. This is the case for $N_f = 1$, or for $N_f = 2$ or $N_f = 3$ with generic and nonvanishing masses. *The physical quark numbers of these massless monopoles turn out to be zero.*

To see how this result comes about, let us consider the one-flavor case and concentrate on the $(n_m, n_e, n_1) = (1, 0, 0)$ monopole occurring at the singularity $u = u_3 = e^{-i\pi/3}$. (We use the unit, $3 \cdot 2^{-8/3}\,\Lambda^2 = 1$.) The proof in other cases is similar. From eq. (4.7) one has

$$\tau_{01} = -\frac{1}{2\pi}\left(\oint_\alpha \frac{dx}{xy} - \tau \oint_\beta \frac{dx}{xy}\right); \quad \tau = \frac{da_D}{da} = \oint_\alpha \frac{dx}{y} \Big/ \oint_\beta \frac{dx}{y}, \tag{7.1}$$

where use of made of the explicit formulae valid for $N_f = 1$, $\lambda_{SW} = -(\sqrt{2}/4\pi)y\, dx/x^2$; $y = x^2(x-u) + 2\sqrt{2}a_1 x - 1$. Near $u \simeq u_3$, two of the branch points x_2, x_3 are close to each other (and coalesce at $2u_3/3$ when $u = u_3$), while the third one is near $-u_3/3$. In the integrations over α cycle (which encircles the nearby branch points x_2 and x_3) y can be approximated as $y = \sqrt{(x-x_1)(x-x_2)(x-x_3)} \simeq \sqrt{x_2 - x_1}\sqrt{(x-x_2)(x-x_3)}$, so that

$$\oint_\alpha \frac{dx}{xy} = 2\int_{x_3}^{x_2} \frac{dx}{xy} \to \frac{2\pi}{x_2\sqrt{x_2 - x_1}}, \qquad \oint_\alpha \frac{dx}{y} \to \frac{2\pi}{\sqrt{x_2 - x_1}}, \tag{7.2}$$

as $u \to u_3$. On the other hand, the integration over β cycle is also dominated by the region near $x \simeq x_2$, and

$$\oint_\beta \frac{dx}{xy} = 2\int_{x_2}^{x_1} \frac{dx}{xy} \simeq \frac{2}{x_2\sqrt{x_2 - x_1}} \cdot I\,, \tag{7.3}$$

where the integral $I = \int_{x_2}^{x_1} \frac{dx}{\sqrt{(x-x_2)(x-x_3)}}$ is divergent at $u = u_3$. However the $\oint_\beta \frac{dx}{y}$ in the denominator of τ also diverges as

$$\oint_\beta \frac{dx}{y} \simeq \frac{2}{\sqrt{x_2 - x_1}} \cdot I\,. \tag{7.4}$$

[4] In [4] the present authors studied the electric and quark number fractionalization in the context of the original $SU(2)$ gauge theory with N_f quark hypermultiplets. For the quark number charges, it was only possible to make a check through the mass formula, which is known both semiclassically and exactly [2].

Therefore

$$\tau_{01} \to -\{\frac{1}{x_2\sqrt{x_2-x_1}} - \frac{1}{x_2\sqrt{x_2-x_1}} \cdot \frac{I}{I}\} = 0, \quad S^{(phys)} \to 0\,, \tag{7.5}$$

as $u \to u_3$. The same result follows by using the second formula eq. (5.2).

For the massless $(1,1,1)$ dyon at $u = u_1 = e^{+i\pi/3}$ (where the combined $\alpha+\beta$ cycle vanishes) one finds by a similar analysis that $S^{(phys)} = 1 - 1 = 0$.

These results are somewhat analogous to the fact that all massless "dyons" of $N = 2$ Seiberg-Witten theories with various N_f, are actually all pure magnetic monopoles with $Q_e = 0$. [4]

8. Some numerical results

It is straightforward to evaluate τ_{0i} given by the formula (4.7) or (5.2) numerically at any point on the moduli space. In Fig.1 we show $\mathrm{Re}(\tau_{01})$, or the physical quark number of $(1,0,0)$ dyon in N=2 SQCD with a single massless flavor. In this case $S^{(phys)}$ approaches $-1/2$ in the weak coupling limit, while it rapidly reduced to zero near the singularity where the $(1,0,0)$ BPS state becomes massless, in accordance with the discussion in the precedent paragraph. The quark number of the same dyon remains equal to $-1/2$ in any vacua with real positive u.

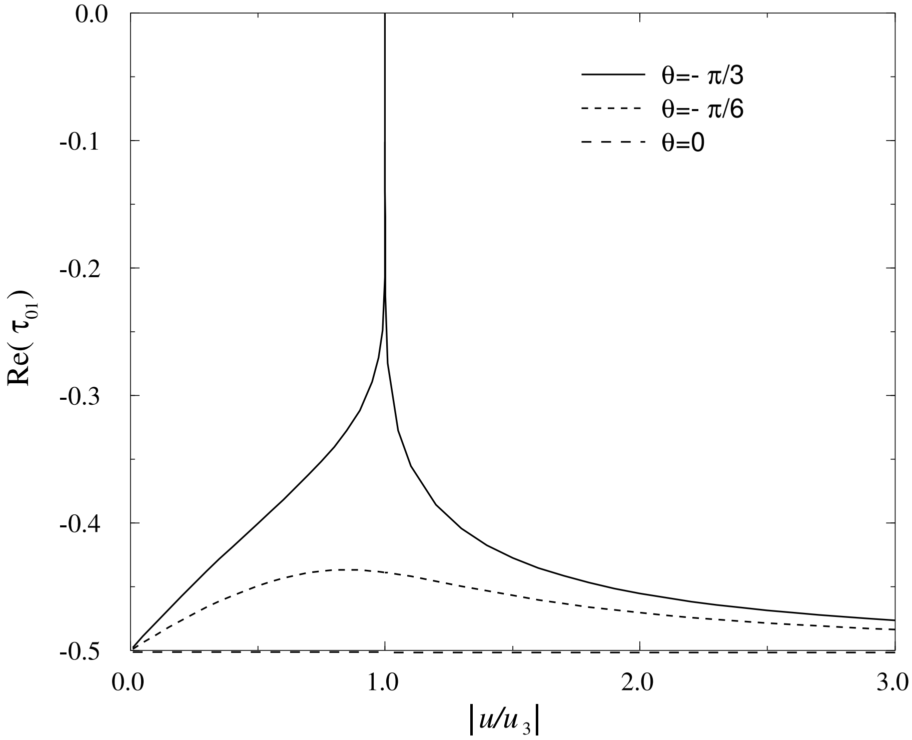

Figure 1: $\mathrm{Re}(\tau_{01})$ in the massless $N_f = 1$ theory is shown along the half lines $u = |u| \exp i\theta$ of $\theta = 0, -\pi/6, -\pi/3$.

CP invariance and quark numbers

It is somewhat surprising that the physical quark number of the monopole takes all possible fractional values even in a CP invariant theory. An example is the $N_f = 1$, $m = 0$ theory at $\mathrm{Arg}\, u = -\pi/3$; $|u| \geq 1$, where $\theta_{eff} = 0$. Indeed $S^{(phys)}$ of the $(1,0)$ monopole takes all real values from 0 (at $u = e^{-i\pi/3}$) to $-1/2$ (at $|u| \to \infty$). Such a result seems to be at odd with the well-known result of Jackiw and Rebbi [8], that in a CP invariant $SU(2)$ theory with a fermion in the fundamental representation the 't Hooft-Polyakov monopole becomes a degenerate doublet with fermion numbers $q_\pm = \pm 1/2$.

The key for solving this apparent puzzle lies in the vacuum degeneracy. In the argument of Ref. [8] the standard monopole $|0\rangle \equiv |M_-\rangle$ is accompanied by another state

$$|M_+\rangle = b^\dagger |M_-\rangle, \quad \text{such that} \quad \langle M_-|\psi|M_+\rangle \neq 0\,, \tag{8.1}$$

where b is the fermion zero mode operator, $\psi = b\,\psi_0(x) + \text{nonzero modes}$, $\{b, b^\dagger\} = 1$. There is a conserved fermion conjugation symmetry $\mathcal{F}$, such that

$$[\mathcal{F}, H] = 0\,; \quad \mathcal{F}\, b\, \mathcal{F} = b^\dagger, \quad \mathcal{F}^2 = \mathbf{1}\,. \tag{8.2}$$

This last equation, together with eq. (8.1), implies that the monopole states $|M_\pm\rangle$ transform to each other by $\mathcal{F}$: $\mathcal{F}|M_\pm\rangle = |M_\mp\rangle$. The fermion number operator must be defined as

$$S = \frac{1}{2}\int d^4x\, (\psi^\dagger\psi - \psi\psi^\dagger) = \frac{1}{2}(b^\dagger b - b b^\dagger) + \ldots \tag{8.3}$$

so that

$$\mathcal{F}S\mathcal{F} = -S, \quad [S, \psi] = -\psi\,. \tag{8.4}$$

One finds $q_+ = -q_-$ from the first of the above. On the other hand, from the $\langle M_-|\ldots|M_+\rangle$ matrix element of the second of eq. (8.4) another relation, $q_+ = q_- + 1$ follows. Combining these two one finds the announced result $q_+ = -q_- = 1/2$. Of course, the first of eq. (8.2) guarantees that the states $|M_\pm\rangle$ are degenerate in mass.

In the present theory the role of the fermion conjugation is played by CP symmetry. Under CP, however, the vacuum is also transformed as $u \to u^*$. What happens in the case of theories at $\mathrm{Arg}\, u = -\pi/3$; $|u| \geq 1$, is that the theory is transformed by CP to another theory, related to the original one by an exact Z_3 symmetry. This explains the fact that $\theta_{eff} = 0$ and that the low energy effective monopole theory there has an exact CP invariance in the usual sense [4]. Nonetheless, from the formal point of view the original CP symmetry of the underlying theory is spontaneously broken in this case, and the Jackiw-Rebbi argument does not apply. In fact, although the operator relations eq. (8.2) and eq. (8.4) still hold, the states are now transformed by $\mathcal{F}|M_\pm; u\rangle = |M_\mp; u^*\rangle$: the two states related by CP operation live on two different Hilbert spaces. As a result, the first of eq. (8.4) yields $q_+^* = -q_-$; $\quad q_+ = -q_-^*$, where $q_\pm^*$ are the quark number of the states $|M_\pm; u^*\rangle$. The second equation, whose matrix elements relate the states in the same vacuum, leads to $q_- = q_+ - 1$; $\quad q_-^* = q_+^* - 1$. Note that these four relations are mutually consistent and relates the four charges by $q_+^* = -q_- = -q_+ + 1 = q_-^* + 1$. Although the first of eq. (8.2) does imply that the monopoles $|M_+; u^*\rangle$ and $|M_-; u\rangle$ have the same mass, (and similarly $|M_-; u^*\rangle$ and $|M_+; u\rangle$) it does not imply any degeneracy; it rather means that the spectrum of the theories at u and at u^* are the same, reflecting the Z_3 symmetry of the underlying theory .

Note that along the real positive values of u (for $N_f = 1$), where CP is exact and not spontaneously broken (with a CP invariant vacuum), dyons are found indeed to be doubly degenerate and have quark numbers $\pm 1/2$, in accordance with [8].

9. Quark-number current correlation functions

Once the physical quark numbers of light dyons are known, the analogue of the R-ratio associated with the correlation function eq. (1.1) may be computed at low energies by the one loop contributions of the weakly coupled dyons. By an appropriate normalization one finds that, near a nondegenerate singularity, the light monopoles and the fermion partners ψ_M, $\tilde{\psi}_M$, $M, \tilde{M}$, add up to give $Disc_{Q^2}\Pi(Q^2) \simeq 3\,(S^{(phys)})^2$, for $Q^2 \ll \Lambda^2$, while at high energies quarks and squarks yield $Disc_{Q^2}\Pi(Q^2) \simeq N_c\,(1 + 1 + 2 \cdot (1/2)) = 6$.

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